Fractal Polytopes

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1 Abstract

This paper introduces and studies the properties of fractal polytopes, fractals formed by repeatedly replacing a polytope with smaller polytopes at its vertices. This family of fractals contains many classical fractal constructions, such as the Sierpinski gasket and the Cantor set. For every regular polytope, we show how to construct a just-touching regular fractal polytope. Some irregular polytopes can be made into just-touching fractal polytopes, and we prove some preliminary results towards determining which polytopes admit a just-touching fractal polytope. Finally, we show the significance of fractal polytopes in determining the structure of connectivity loci for Iterated Function System families.

2 Introduction

A polytope is the n-dimensional generalization of a polygon or polyhedra, a hyper-polyhedra, if you will. We informally define the *fractal polytope*, denoted $F(P,R)$, for P a polytope and with N vertices and $0 \leq R < 1$, to be the limit of the construction which takes P and replaces it with N smaller P's, each with edge length R times that of P, and placed at P's vertices. (See Figure 1).

This construction can be made more precise by defining it as an iterated function system. As defined by Barnsley [Barnsley 88], an iterated function system (IFS) is a set of contractive transformations $W = \{w_i\}$ on a complete metric space (for our purposes, this will be $\Re^n$ under the Euclidean metric). The main property of an IFS is that there is an unique compact subset of the space which is invariant under W, called W's attractor; i.e., a set A (or sometimes A_W) for which $A = W(A) = \bigcup w_i(A)$.

Definition 1 *$F(P,R)$ is the attractor of the IFS $\{w_i : \Re^n \rightarrow \Re^n \mid w_i(z) = Rz + v_i(1-R),\ v_i = i^{th}$ vertex of $P,\ i = 1 \ldots N\}$.*

By construction, F(P,R) is easily seen to have similarity dimension $\log(N)/\log(1/R)$, and thus deserves the name fractal.

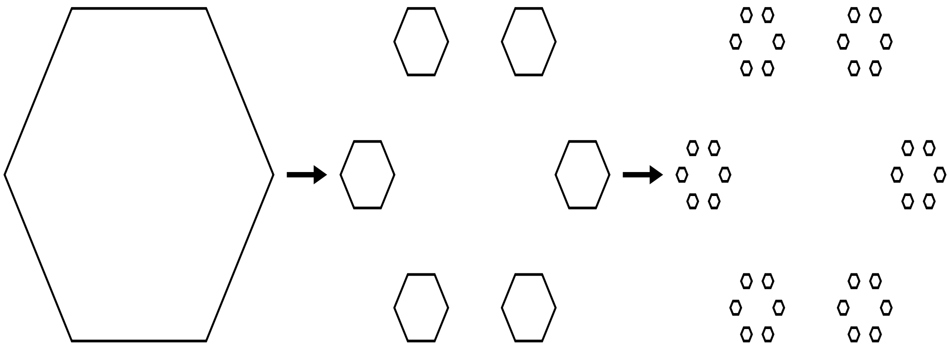

Figure 1: First two stages in the construction of a fractal polytope

Why do we define fractal polytopes to be made up of transformations towards the vertices of the polytope, rather than, say, the edges or faces? First, our definition generalizes many "classical" fractals: F(unit interval,1/3) is the terninary Cantor set, F(triangle,1/2) the Sierpinski gasket, and the boundary (and interior) of F(hexagon,1/3) is a Koch snowflake. Second, the IFS made up of shrinking the polytope towards its edges (or faces, etc.) is exactly isomorphic to the fractal polytope of the polytope's edge (or face, etc.) dual. For instance, shrinking a cube towards its faces gives the fractal polytope of an octahedron, the cube's face dual. Similarly, adding transformations to the fractal, say on the outside of the faces (as does [Wagon 91]), just makes it equivalent to a fractal polytope with added vertices. The third and real reason is that contracting the polytope towards its vertices allows us to retain a large amount of the geometry of the situation, in a sense that Theorem 1 will make precise.

3 Just-Touching Regular Polytopes

For our purposes, an IFS is said to be disconnected if $w_i(A)\bigcap w_j(A) = \emptyset\ \forall i,j$. An IFS is just touching if it is not disconnected and if its attractor contains an open set ϑ such that $w_i(\vartheta)\bigcap w_j(\vartheta) = \emptyset\ \forall i,j : i \neq j$ and $W(\vartheta) = \bigcup w_i(\vartheta) \subset \vartheta$. An IFS which is not disconnected or just touching is called overlapping.

Note that an IFS being disconnected or not implies nothing about its attractor being disconnected or not, though for all the fractal polytopes in this section, these two notions will be equivalent.

Theorem 1 *If P is a regular polytope and v_1 and v_2 are two neighboring vertices, then F(P,R) is disconnected iff $w_1(P)$ and $w_2(P)$ are disconnected and just-touching if $w_1(P)\bigcap w_2(P) \neq \emptyset$ has measure 0.*

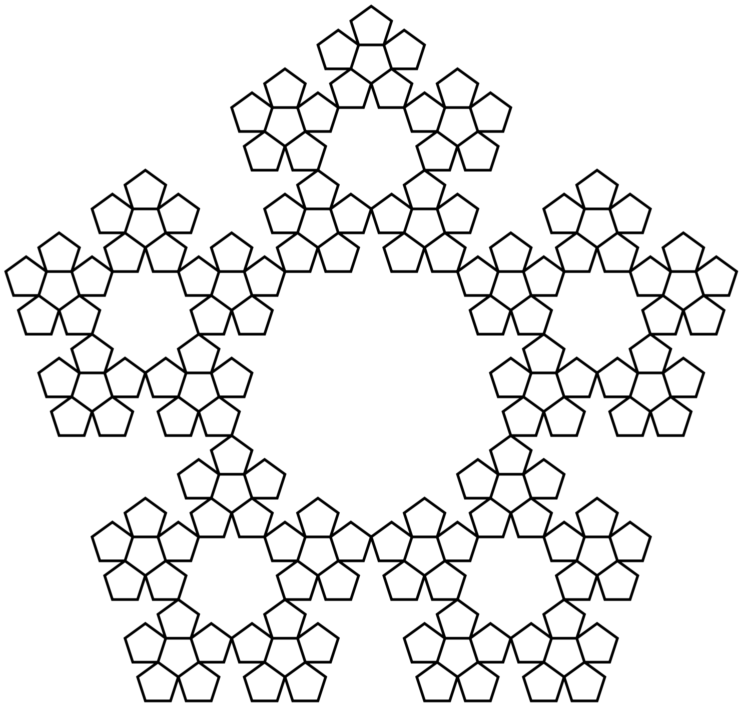

Figure 2: A just-touching fractal pentagon ($R = 0.381971$)

Proof: Since P is convex, $A = F(P,R) \subset P$, and we note that if none of the $w_i(P)$'s intersect, then none of the $w_i(A)$'s do either, so F(P,R) is disconnected. Since P is regular, there is an isomorphism of P which takes any edge to any other edge, so if $w_i(P)$ and $w_j(P)$ intersect, then so do $w_1(P)$ and $w_2(P)$.

The next thing that we must show is that if $w_1(P) \bigcap w_2(P)$ has measure 0, then F(P,R) just-touches. We claim that a just-touching intersection of $w_1(P)$ and $w_2(P)$ implies the intersection of two vertices on $w_1(P)$ and $w_2(P)$. This is because a regular polytope has a symmetry which swaps two adjacent vertices. This symmetry implies that any intersection between $w_1(P)$ and $w_2(P)$ must be on the hyper-plane which bisects the edge between v_1 and v_2. Since the intersection of a convex polytope and a hyper-plane must include a vertex, the intersection between $w_1(P)$ and $w_2(P)$ must include the intersection of two vertices, say $w_1(v_i) = w_2(v_j)$.

But the vertices of P are fixed points of the transformations of F(P,R)'s IFS, so they, along with all of their images, are on the attractor A. In particular, $w_1(v_i)$ and $w_2(v_j)$ are on the attractor. And since the $w_i(P)$'s are connected by points on the attractor, A is connected by induction. It is clear that A minus the (n-1)-dimensional faces of P satisfies the open-set condition for being just-touching.

Corollary 1 *If P is a regular polytope, there exists an unique R such that F(P,R) is just touching, and for which all $F(P,R')$, $R' < R$ is disconnected.*

Proof: The distance between the closest two points of $w_1(P)$ and $w_2(P)$ is a non-decreasing function of R. There is thus an unique R for which $w_1(P)$ and $w_2(P)$ are just-touching, and for which for all $R' < R$, $w_1(P)$ and $w_2(P)$ are disconnected.

Theorem 1 allows us to more or less ignore the fractal structure of the fractal

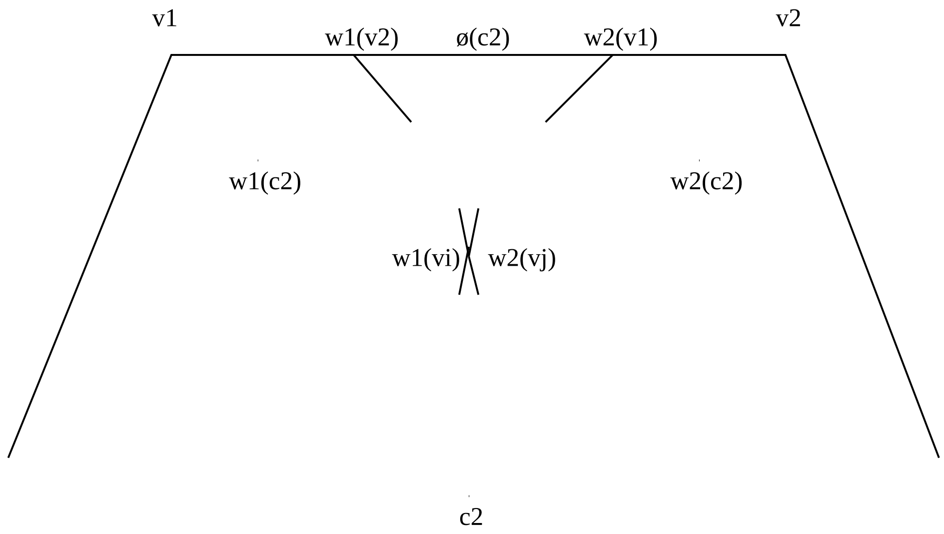

Figure 3: When $w_1(P)$ and $w_2(P)$ just-touch, $R = \overline{v_1 w_1(v_2)}/\overline{v_1 v_2} = 1/(1 + 2\phi(w_1(c_2))\phi(c_2))$

polytope, and focus on the good old fashioned geometry of the intersection between $w_1(P)$ and $w_2(P)$. Theorems 2 - 5 use this fact to classify all regular fractal polytopes.

Lemma 1 *If m is the length of a regular polytope P with unit edge length orthogonally projected onto a line containing an edge of P, then $F(P, 1/(1+m))$ is just touching.*

Proof: (See Figure 3). Let c_2 be the center of P, $\overline{ab}$ be the distance between points a and b, and ϕ be the projection of P onto the edge $v_1 v_2$. Then, $\overline{v_1 v_2} = \overline{v_1 \phi(w_1(c_2))} + \overline{\phi(w_1(c_2)\phi(c_2)} + \overline{\phi(c_2)\phi(w_2(c_2))} + \overline{\phi(w_2(c_2))v_2}$. But $\overline{v_1 \phi(w_1(c_2))} = \overline{\phi(w_2(c_2))v_2} = 1/2R\overline{v_1 v_2}$. Also, by Theorem 1, when F(P,R) is just-touching, $w_1(P)$ and $w_2(P)$ just-touch on the hyperplane which ϕ projects down to $\phi(c_2)$. By hypothesis, $\phi(w_1(P))$ has length $mR\overline{v_1 v_2}$, so $\overline{\phi(w_1(c_2))\phi(c_2)} = \overline{\phi(c_2)\phi(w_2(c_2))} = 1/2mR\overline{v_1 v_2}$. Thus $\overline{v_1 v_2} = R\overline{v_1 v_2} + mR\overline{v_1 v_2}$, and $R = 1/(1+m)$.

Theorem 2 *[Colthurst 89], [DiDomizio 90] $F($ Regular N-gon, $1/(1+\sin((2\lceil N/4\rceil - 1)\pi/N)/\sin(\pi/N)))$ is just connected.*

Proof: (See Figure 4). If we label the vertices of the N-gon, in order, as $v_1, \ldots,$ v_N, and project the N-gon onto the edge $v_1 v_2$, then the vertex farthest away from the center, in this projection, will be $v_{\lceil N/4\rceil+1}$. In fact, it will be $\sin((2(\lceil N/4\rceil + 1) - 3)\pi/N)\overline{v_1 c_2}$ away from the center. And since, $\overline{v_1 c_2} = 1/2\overline{v_1 v_2}/sin(\pi/N)$, $m = \sin((2\lceil N/4\rceil - 1)\pi/N)\csc(\pi/N)$, and we have by Lemma 1 that $R = 1/(1+\sin((2\lceil N/4\rceil - 1)\pi/N)/\sin(\pi/N))$.

Theorem 2's formula for R can be easily simplified for the various possibilities of $\lceil n/4\rceil$. For instance, $R = 1/(1+cot(\pi/N))$ for $N \equiv 0 \pmod 4$. The value

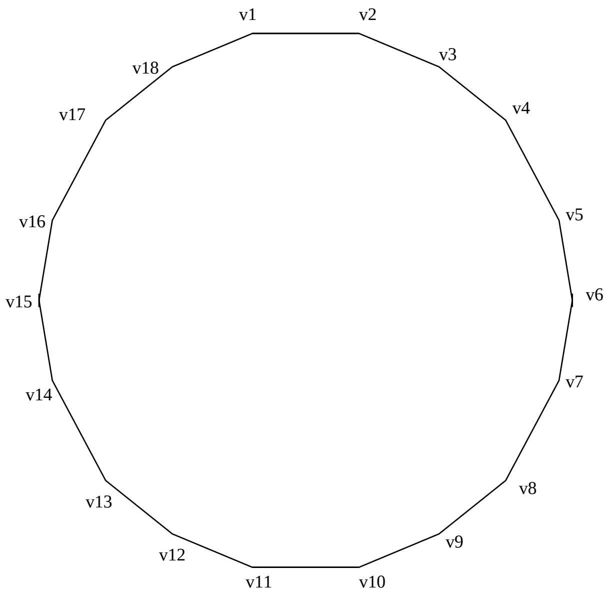

Figure 4: If you squash a regular N-gon, the $\lceil N/4 \rceil + 1^{th}$ vertex sticks out.

of $N \pmod 4$ also determines how the fractal N-gon just touches – at an edge if $N \equiv 0 \pmod 4$, and at a vertex otherwise.

Theorem 3 *F(Dodecahedron, $1/(1+\tau^2)$) and F(Icosahedron, $1/(1+\tau)$) are just connected.* [1]

Proof: . If we place our smaller icosahedron into the vertices of a larger one, we see that they meet at edges. Another way of seeing this is to project the icosahedron onto the plane containing an edge and the center, which gives an irregular hexagon. The vertex of the hexagon which just-touches its neighbors is the projection of an edge of the icosahedron.

Similarly, the projection of a dodecahedron onto a plane containing an edge and the center is an irregular hexagon, the touching vertex of which is again the projection of an edge. (The orthogonal projection we are using is called an edge first projection by Coexter. See [Coexter 73, page 241]).

Using the fact ([Coexter 73, page 292]) that the distance from the center of an icosahedron to the center of an edge is $1/2\tau$ times the edge length, we calculate that $R = 1/(1+\tau)$. Similarly, for the dodecahedron, the center-midpoint to edge length ratio is $1/2\tau^2$, so $R = 1/(1+\tau^2)$.

The following regular polytopes occur in every dimension, and we attack them all in one fell swoop.

Theorem 4 *F(Hyper-Cube, 1/2), F(Simplex, 1/2), F(Cross-Polytope, 1/2) are all just connected, in any dimension n.*

Proof: F(hyper-cube, 1/2) is the original hyper-cube. If we assign its vertices to be $(\pm 1, \ldots, \pm 1)$, then w_i(hyper-cube) is the unit hyper-cube in the i^{th}

[1] $\tau = 1/2(\sqrt{5}+1)$, the golden ratio

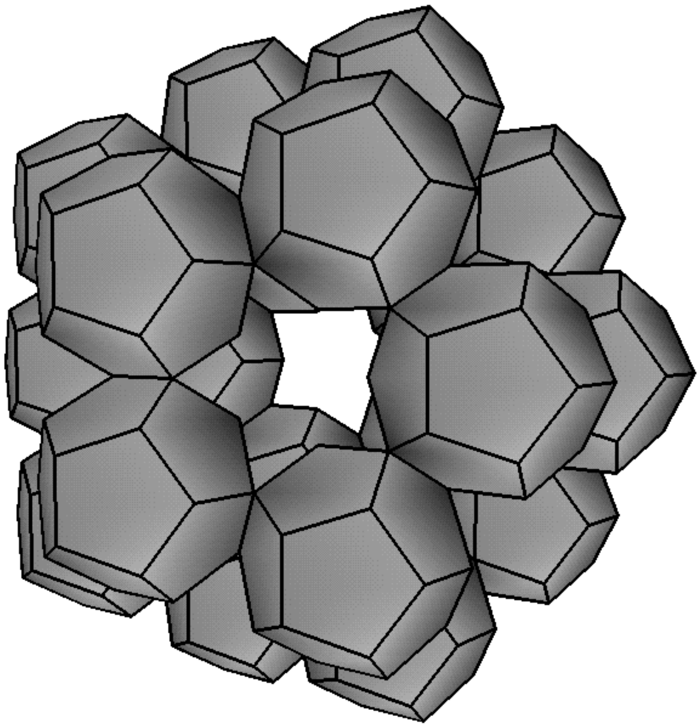

Figure 5: First stage in a just-touching fractal dodecahedron

quadrant. Each of these touches its neighbors on a $n-1$ dimensional hypercube face, and all of them touch at the origin.

If we assign the vertices of F(cross-polytope, 1/2) to be the permutations of $(\pm 1, 0, \ldots, 0)$, then each w_i(cross-polytope) touches its neighbors on the edge from the origin to a midpoint of the original cross-polytope, and touches all of the w_j(cross-polytope)'s at the origin.

The $n-1$ dimensional face of F(cross-polytope, 1/2) is the $n-1$ dimensional F(simplex, 1/2). Each w_i(simplex, 1/2) touches all of the others at a vertex on a midpoint of the original simplex.

The reader may not be aware that in dimensions five and greater, the above polytopes – the hypercube, the simplex, and the cross-polytope – are the only regular polytopes. This I feel is one of the saddest results in mathematics (a proof can be found in [Coexter 73, pages 133-136]). It does, however, allow us to complete our classification with these four dimensional polytopes:

Theorem 5 *F(24-Cell, $1/3$), F(120-Cell, $1/(1+\tau^4)$), and F(600-Cell, $1/(1+2\tau)$) are all just connected.*

Proof: Our technique, as in Theorem 3 above, will be to project the polytope down onto a plane containing an edge and the center of the polytope.

Performing this projection on a 24-Cell gives a regular hexagon. You can see this by taking its vertices to be $(\pm 1, \pm 1, 0, 0)$, and then projecting it orthogonally

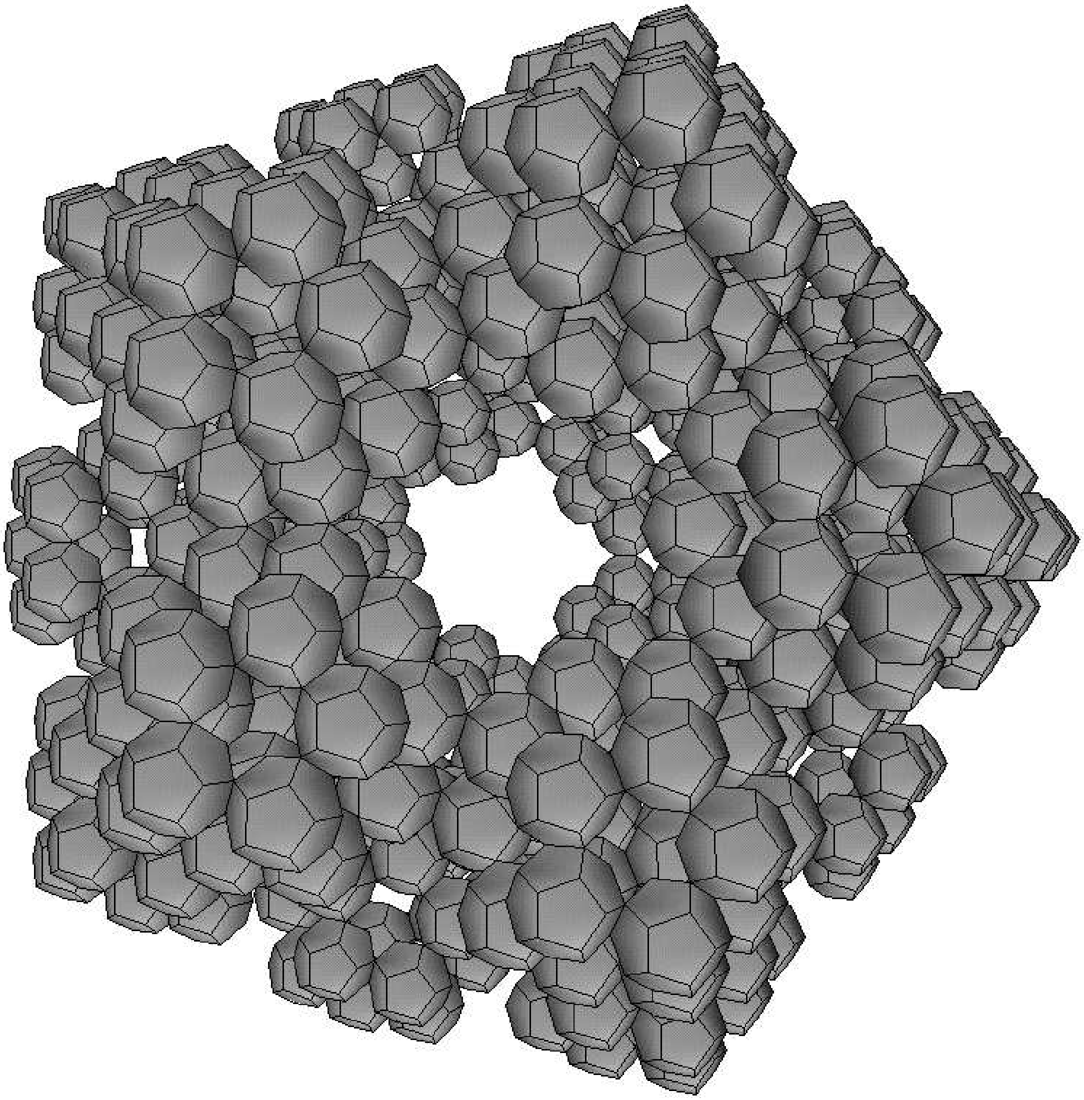

Figure 6: Second stage in a just-touching fractal dodecahedron

into three space by dropping the last coordinate. This gives a cuboctahedron (with two extra vertices on the center of each square face), which if we project onto a plane containing a proper edge and the center, gives us a hexagon (Compare with [Coexter 73, page 242, figure 12]). A single vertex projects down onto the vertex of the hexagon which touches its neighbors, so the 24-Cell (and the cuboctahedron) just-touches at a vertex as well. Since the distance from the center of the 24-Cell to one of its vertices is the same as its edge length, the 24-Cell projects down onto a regular hexgaon, and like a regular hexagon, it has a just-touching factor of 1/3.

The 600-cell and the 120-cell have so many vertices that we will stop trying to visualize each stage in the projection, and will instead just work with the coordinates. The 600-cell has 120 vertices, which we can represent (with an edge length of $2\tau^{-1}$ as the eight permutations of $(\pm 2, 0, 0, 0)$, the sixteen permutations of $(\pm 1, \pm 1, \pm 1, \pm 1)$, and the 96 even permutations of $(\pm\tau, \pm 1, \pm\tau^{-1}, 0)$ [Coexter 73, pages 156-157]. If we take our selected edge to be $(\tau, 1, \pm tau^{-1}, 0)$ and perform our projection by dropping the last coordinate, we will note that the most extreme vertices in the z direction are $(0, 0, \pm 2, 0)$. The 600-cell thus just-touches at a vertex directly halfway inbetween $w_1(c_2)$ and $w_2(c_2)$. The center-vertex to edge length ratio is τ, so $R = 1/(1 + 2\tau)$.

The 600-vertices of a 120-cell of edge length $2\tau^{-2}$ can be taken to be the permutations of $(\pm 2,\ \pm 2,\ 0,\ 0)$, $(\pm\sqrt{5},\ \pm 1,\ \pm 1,\ \pm 1\ )$, $(\pm\tau,\ \pm\tau,\ \pm\tau,\ \pm\tau^{-2}\ )$, $(\pm\tau^2,\ \pm\tau^{-1},\ \pm\tau^{-1},\ \pm\tau^{-1}\ )$, and the even permutations of $(\pm\tau^2,\ \pm\tau^{-2},\ \pm 1,\ 0\ )$, $(\pm\sqrt{5},\ \pm\tau^{-1},\ \pm\tau,\ 0\ )$, $(\pm 2,\ \pm 1,\ \pm\tau,\ \pm\tau^{-1}\ )$ [Coexter 73, pages 156-157]. If we select $(\tau^2, \pm\tau^{-2}, 1, 0)$ to be the relevant edge, and perform the projection by dropping the last coordinate, we will note that the most extreme vertices in the y-coordinate are those on the plane $y = \tau^2$. There are twelve of these: $(\pm\tau^{-1}, \tau^2, \pm\tau^{-1})$, $(\pm\tau^{-2}, \tau^2, 0)$, $(\pm 1, \tau^2, \pm\tau^{-2})$, and $(0, \tau^2, \pm 1)$, which are the projection of the twenty vertices of a dodecahedron face of the 120-cell! The dodecahedron-center to edge length ratio is $1/2\tau^4$, so $R = 1/(1 + \tau^4)$.

4 Irregular Fractal Polytopes

We have already encountered one irregular fractal polytope, the fractal cuboctahedron obtained by projecting a 24-Cell (Actually, we encountered a cuboctahedron with extra vertices on its faces, but the real fractal cuboctahedron does have a just-touching scaling factor of 1/3, like the hexagon and 24-Cell). The cuboctahedron is special in the sense that it is quasi-regular: it has only two types of faces, and each face is directly surrounded by faces of the other type. The other convex quasi-regular polyhedron [Coexter 73, page 18], the icosidodecahedron, also admits a just-touching fractal polytope, for $R = 1/(1 + csc(\pi/10))$.

The convex quasi-regular polyhedron posses the nice property that there is an isomorphism of the polyhedron which takes any edge to any other edge, and an isomorphism which switches the two vertices of an edge. Let us call any polytope quasi-regular if it has this property. It easily follows that a n-dimensional quasi-regular polytope has at most two different types of (n-1)-dimensional faces, each of which must be itself quasi-regular. Note that quasi-regularity is precisely the symmetry condition used in Theorem 1. When this symmetry condition does not hold, we are not guaranteed a just-touching fractal polytope. For instance, for the truncated cube, for $R = 1/(1 + cot(\pi/12))$, adjacent vertices on edges on the triangular faces have truncated cubes which just touch, but not vertices on edges on the octagonal faces (which just touch for $R = 1/(1 + cot(\pi/8))$.

In the general case, then, we will have a scaling ratio for each different type of edge. These scaling ratios, however, need not be different – we can still have just-touching fractal polytopes. For instance, for a triangular prism, the scaling ratio for vertices on edges on the triangular faces is 1/2 and so is the scaling ratio for the vertices on edges on the square faces. Other examples of just touching irregular fractal polyhedra are F(Truncated Tetrahedron, 1/3), F(Truncated Octahedron, 1/4), and F(Rhombicoboctahedron, $1/(1 + cot(\pi/8))$).

The following theorem gives a sufficient, but not necessary, condition for a polytope to admit a just-touching fractal. Figure 7 gives an example of a just-touching fractal polytope which is not covered by this theorem. It remains an open problem to give a characterization of all just-touching fractal polytopes.

Theorem 6 *If P is the affine transformation of a quasi-regular polytope, then there is an unique R such that $F(P, R)$ is just-touching.*

Proof: The proof is the same as in Theorem 1, with the added observations that a quasi-regular polytope is by necessity convex, and that since affine transformations commute with scalings, the affine transformation of a fractal polytope is the fractalization of the affine transformed polytope.

5 Complex Fractal Polygons

So far, we have been treating the scaling parameter R as a real number. If P is a polygon, then there is a natural interpretation of R as a complex number: giving the shrunken polygon at each vertex a twist. This motivates the following definition of the complex fractal polygon, $\mathbf{C}F(P, R)$:

Definition 2 *$\mathbf{C}F(N - gon, R)$ is the attractor of the IFS $\{w_i : \mathbf{C} \rightarrow \mathbf{C} \mid w_i(z) = Rz + v_i(1 - R),\ v_i = e^{2\pi i/N},\ i = 1 \ldots N\}$.*

There is an incredible variety of complex fractal polygons. Even the simplest case of complex fractal lines is mind-bogglingly diverse. For instance, Knuth's dragon fractal is **C**F(line, (1-i)/2).

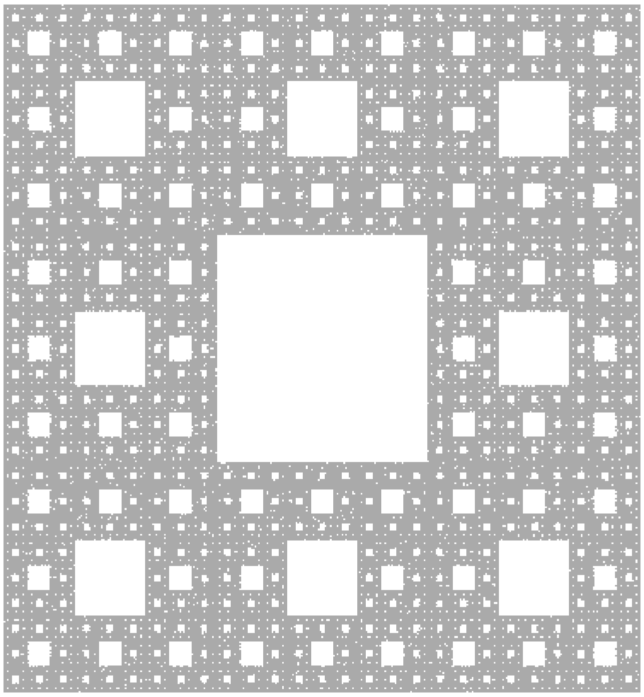

Figure 7: A just-touching fractal which isn't quasi-regular

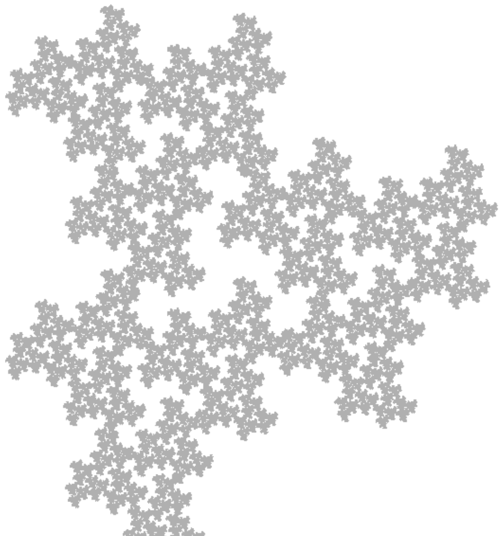

Figure 8: A typical complex fractal polygon – **C**F(Triangle, 0.48 + 0.25 i)

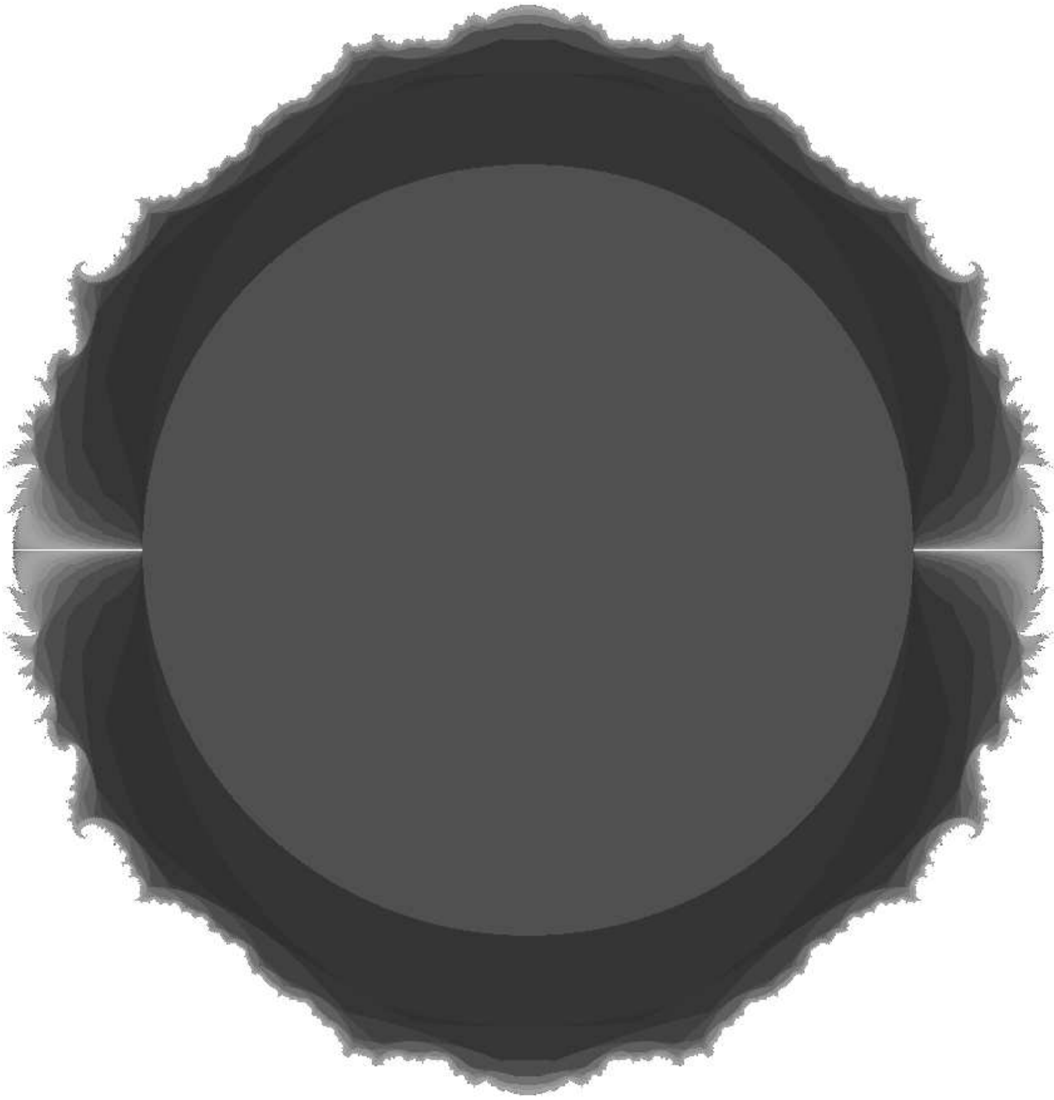

Figure 9: The complement of the Mandelbrot set for **C**F(line,R) $\equiv \{sz \pm 1\}$

Since R is now a complex number, it is no longer the case that there is a single value of R for which **C**F(N-gon,R) is just touching. Instead, we will have a set of R values for which this is so. It is customary to consider the set of R values for which the family of fractals is connected: this is called the connectivity locus or Mandelbrot set. The boundary of this set in general consists of just-touching fractals, but not all just-touching **C**F(N-gon,R)s are on the boundary [**C**F(line, (1-i)/2) is a good example of this].

The Mandelbrot set for **C**F(line,R) is the same as the one for $\{sz \pm 1\}$ which has been studied by Barnsley and others ([Barnsley 88], [Barnsley 85], [Barnsley 89], [Vrscay 86] – but note that Barnsley, et al., define their D to be the set of s for which $\{sz \pm 1\}$ is *disconnected*, while we follow the tradition of the quadratic case by defining our Mandelbrot set to be the set of s for which $\{sz \pm 1\}$ is *connected*). A computer rendering of the complement of the Mandelbrot set (using an optimized version of the algorithm in [Barnsley 88, pages 304-305]) appears in Figure 9.

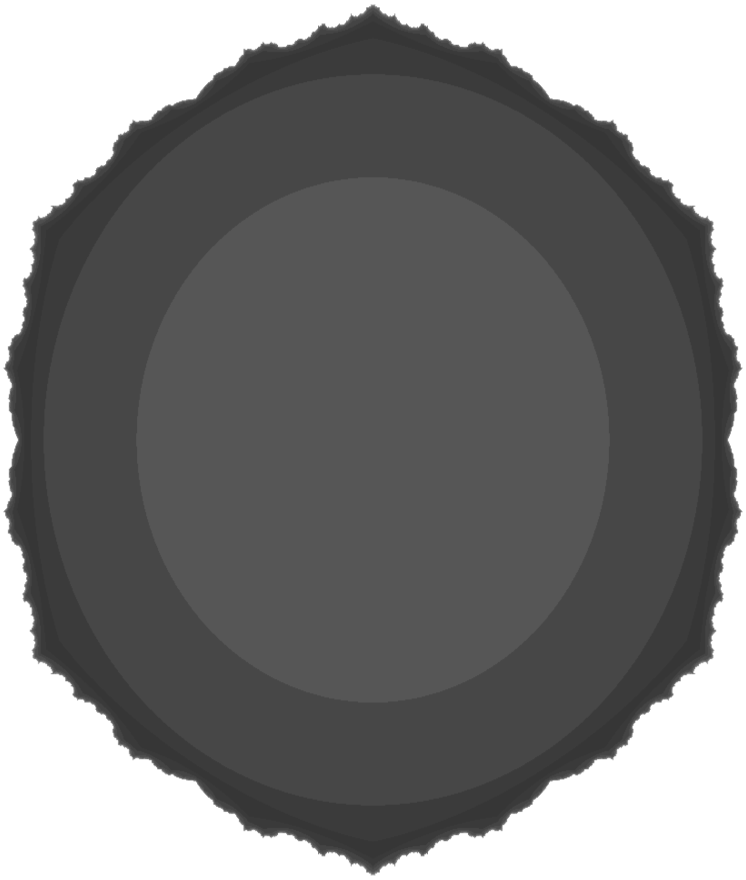

Figure 10: The complement of the Mandelbrot set for **C**F(Equilateral Triangle,R)

Any IFS of two maps, such as any **C**F(line,R), will either be connected or totally disconnected. In general, this will not be so; the attractor of an IFS of more than two maps will usually consist of several connected pieces and an infinite number of disconnected points. The **C**F(N-gon,R)s for regular N-gons are an interesting exception in this regard; because of their symmetry, they are either totally connected or totally disconnected.

Looking at the pictures of the **C**F(N-gon,R)s suggests several natural conclusions:

Theorem 7 $\mathbf{C}F(\text{Regular N-gon}, R)$ *is disconnected iff* $\mathbf{C}F(\text{Regular N-gon}, Re^{2\pi i/N})$ *is disconnected iff* $\mathbf{C}F(\text{Regular N-gon}, \overline{R})$ *is disconnected iff* $\mathbf{C}F(\text{Regular N-gon}, -R)$ *is disconnected.*

Proof: First, taking the complex conjugate of R merely switches the direction of rotation, so $\mathbf{C}F(\text{Regular N-gon}, \overline{R})$ is merely the mirror image of $\mathbf{C}F(\text{Regular N-gon}, R)$. Next, $P = \text{Regular N-gon} = Pe^{2\pi i/N}$, so $\mathbf{C}F(\text{Regular N-gon}, R)$

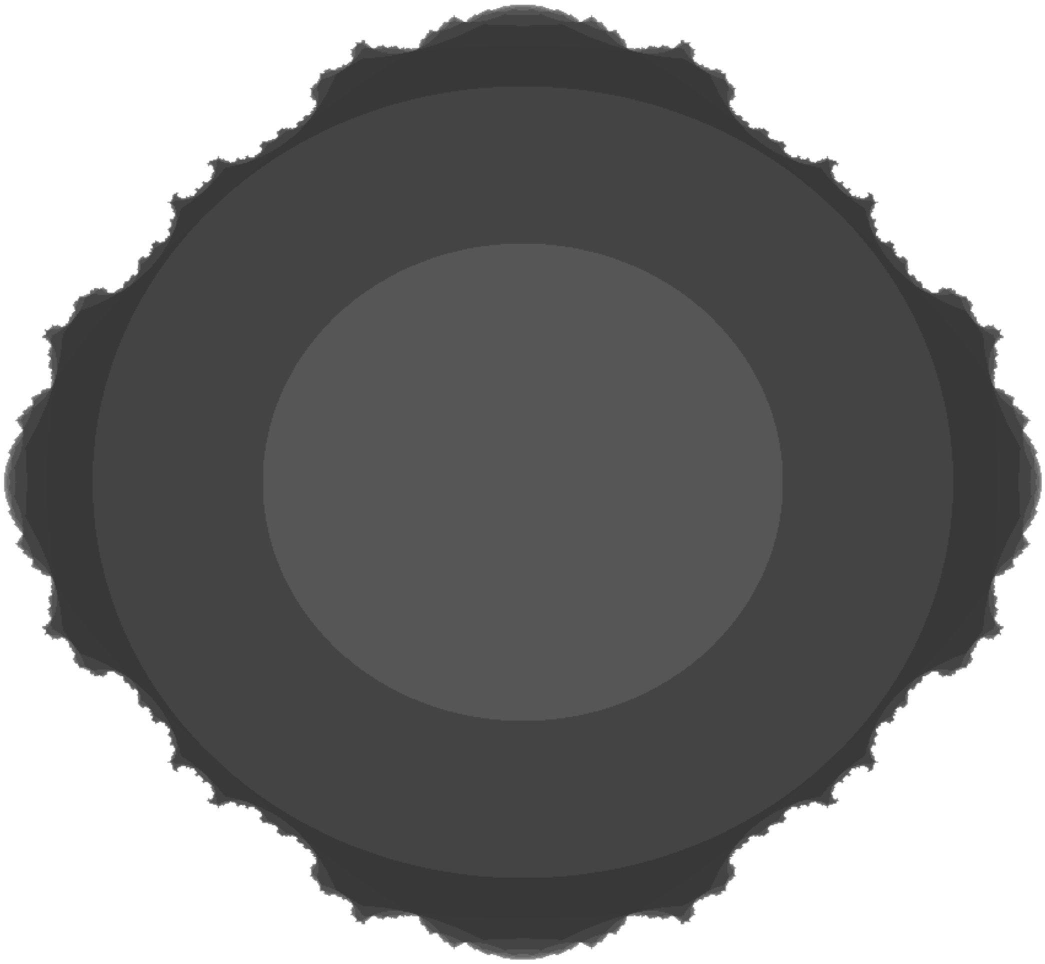

Figure 11: The complement of the Mandelbrot set for **C**F(Square,R)

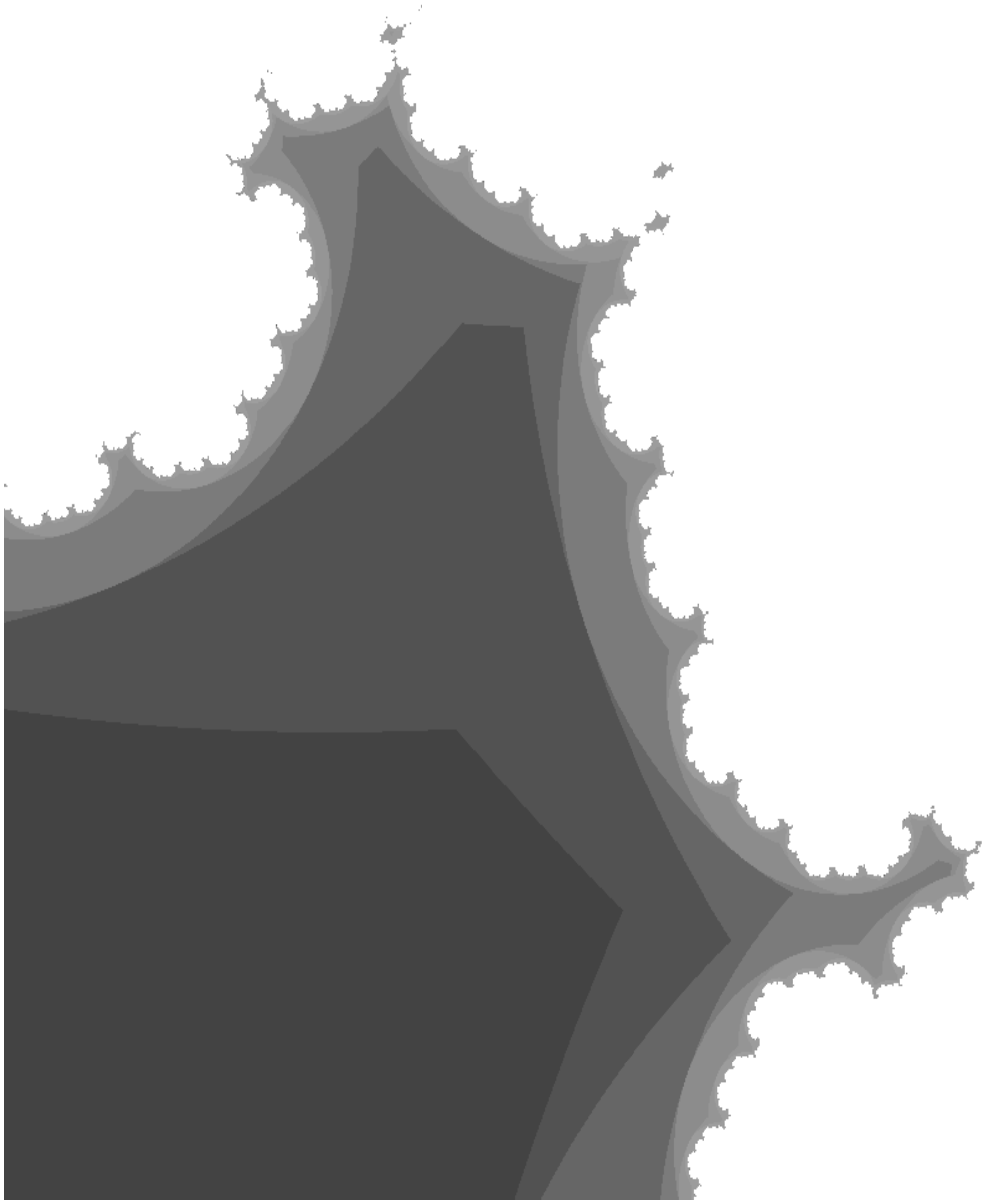

Figure 12: A closeup of the Mandelbrot set for **C**F(Square,R)

is a simple transform of $\mathbf{C}F(\text{Regular N-gon}, Re^{2\pi i/N})$, $z \mapsto z(1-Re^{2\pi i/N})/(1-R)$, in fact. Finally, we note that $\mathbf{C}F(\text{Regular N-gon}, R)$ is equivalent to all sums of the form $\sum_{j=0}^{\infty} a_j R^j$, for all sequences of $a_j \in \sqrt[n]{1}$. To say that $\mathbf{C}F(\text{Regular N-gon}, R)$ is connected is to say that there are sequences $a_j, a'_j \in \sqrt[n]{1}$ such that $\sum_{j=0}^{\infty}(a_j - a'_j)R^j = 0$. But if this is the case then there are clearly sequences $b_j, b'_j \in \sqrt[n]{1}$ such that $\sum_{j=0}^{\infty}(b_j - b'_j)(-R)^j = 0$, so $\mathbf{C}F(\text{Regular N-gon}, R)$ being connected implies $\mathbf{C}F(\text{Regular N-gon}, -R)$ is connected.

Theorem 8 *The boundary of the Mandelbrot set for* $\mathbf{C}$*F(Regular N-gon,R) where it touches the real axis is locally concave if F(N-gon,R) just touches at a vertex.*

Proof: We wish to show that for all small $\epsilon \geq 0$ there is an δ, such that for all $0 < \delta' < \delta$, $\mathbf{C}$F(N-gon, $R + \delta' + \epsilon i$) is disconnected. Let $w_1(v_i) = w_2(v_j)$ be the just touching intersection of F(N-gon,R). The effect of adding *epsiloni* is to rotate $w_1(v_i)$ counterclockwise and $w_1(v_j)$ clockwise, which increases the distance between them. We take ϵ and δ such that $w_1(v_i)$ is the closest point on $w_1(A)$ to $w_2(A)$. It is clear that in this open set, δ is an increasing function of ϵ, and that in this set, $w_1(A)$ and $w_2(A)$ are disconnected.

Theorem 9 *The boundary of the Mandelbrot set for* $\mathbf{C}$*F(Regular N-gon,R) is contained in the annulus* $1/N \leq |R| \leq 1/\sqrt{N}$.

Proof: This proof is a generalization of that in [Barnsley 85] and [Barnsley 88, pages 306-310]. A disconnected fractal must have fractal dimension ≤ 2, while a connected fractal must have fractal dimension ≥ 1. A disconnected or just-touching $\mathbf{C}$F(N-gon,R) is made up of N transformations of scaling factor $|R|$, so it has a fractal dimension of $\log(N)/\log(1/|R|)$. Solving $1 \leq \log(N)/\log(1/|R|) \leq 2$ gives the desired bounds.

Unfortunately, we can not generalize complex fractal polygons to complex fractal polytopes. We could easily define F(P,T) for any polytope P and any contractive affine transformation T, but unless T commuted with all of the symmetries of P, F(P,T) would not retain the symmetries of P, and thus would not retain the desired property of being either totally disconnected or totally connected. But the symmetries of a n dimensional polytope are generated by n-1 linearly independent rotations, and for $n > 2$, the only transformations which commute with n-1 linearly independent rotations are scalar multiples of the identity.

Addendum: Thierry Bousch in his thesis at the Paris de Sud recently proved the following remarkable theorem:

Theorem 10 *The Mandelbrot sets of complex fractal polygons are connected.*

6 Open Questions

Several problems about fractal polytopes remain as yet unanswered.

- If F(P,R) is just touching, is F(P,$R' > R$) overlapping? Any answer to this will have to take account of the geometry of P, since in general, **C**F(P,R) just touching does not imply F(P, sR), $s > 1$, overlapping. More generally, is there any method besides that of fractal dimension estimates to show that a given IFS is overlapping? There are, perhaps, two seperate problems which such a method would have to handle: first to show that the IFS is connected, and second to show that there is some point in the intersection of the attractor's images whose inverse orbit is dense on the attractor.

- It is easy to check that for regular polytopes, $w_i(P)$ and $w_j(P)$ intersecting implies that $w_i(P)$ and $w_k(P)$ intersect, where v_k is a neighbor of v_i. This fact seems to be in general for all convex polytopes – the only counterexamples I have found are for nonconvex polytopes – and should have a simple geometric proof, but I have been unable to find one.

- Find a complete characterization of polytopes which admit a just-touching fractal polytope.

- Knuth's Dragon, **C**F(line,0.5-0.5i), is a just-touching fractal not on the boundary of the Mandelbrot set for **C**F(line,R). Some other fractals with this property are **C**F(triangle,$0.5-i/(2\sqrt{3})$), **C**F(square, $0.5-i\tan(\pi/8)/2$), and **C**F(hexagon,$(3+i)/\sqrt{70}$); all of which, by the way, tile the plane. Are there other just-touching **C**F(N-gon,R)s which are not on the boundary of the Mandelbrot set?

- The Mandelbrot sets for **C**F(Regular N-gon,R), along with those for any family of two transformation IFSs, are natural, in the sense that the associated fractals are either totally connected or totally disconnected, and the Mandelbrot set tells you which. As noted before, this will not be the case in general for an arbitrary family of fractals, which will contain partially connected fractals. Are there any additional natural Mandelbrot sets? [Any family of fractals must be considered up to equivalence of attractors. For instance, the IFS $\{f,g\}$ will have the same attractor as $\{ff,fg,gf,gg\}$ or $\{f,gf,gg\}$. See my paper [Colthurst 92] for a classification of equivalent IFSs.]

7 Acknowledgements

Fractal polygons were first discovered by Andrew Ross and Thomas Colthurst (For this reason, [Ross 90] and [Colthurst 89] refer to them as Colthurst-Ross

Sets and Ross Sets, respectively), and were independently rediscovered by Virginia DiDomizio ([DiDomizio 90]). Both discoveries were very heavily influenced by Michael Barnsley's presentation of the IFS for the Sierpinski gasket as the midpoint game on a triangle.

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